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Dislocations and Internal Length Measurement in Continuized Crystals. I. Riemannian Material Space

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Distributions of dislocations creating point defects are considered. These point defects are described by a metric tensor, which supplements a Burgers field responsible for dislocations. The metric tensor depends on the distribution of dislocations and defines a Riemannian geometry of the material space of a continuized crystal and thus an internal length measurement in this crystal. The dependence of the distribution of dislocations on the existence of point defects created by these dislocations is modeled by treating the Burgers field as a field defined on the Riemannian material space. Field equations, following from geometric identities, are formulated as balance equations on this Riemannian space and their source terms, responsible for interactions of dislocations and point defects, are identified.

1. INTRODUCTION

It is known that the occurrence of many dislocations in a crystalline solid is accompanied by the appearance of point defects. This may be due, for example, to intersections of the dislocation lines; for example, two intersecting right (or left) screw dislocations produce a line of interstitials, and if one screw is right and the other left, a line of vacancies is formed (Frank and Steeds, 1975). The point defects are essentially described by a metric tensor which supplements the torsion tensor of a teleparallel connection responsible for dislocations (Kröner, 1990; Trzęsowski, 1987). On the other hand, dislocations have no influence on local metric properties of the crystal structure of the solid (Kröner, 1985). It is usually described by assuming that the above-mentioned metric tensor is covariantly constant with respect to the teleparallel connection (Trzęsowski, 1987). If the body

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considered is a (three-dimensional) connected manifold $\mathscr{B}$, then the teleparallel connections on it are in one-to-one correspondence with globally defined vectorial moving frames (Sikorski, 1972). If $\Phi = (\mathbf{E}_a; a = 1, 2, 3)$ denotes a global moving frame and $\nabla^\Phi = (\Gamma^A_{BC}[\Phi])$ is the teleparallel covariant derivative (with the connection coefficients $\Gamma^A_{BC}[\Phi]$) corresponding to Φ, then this correspondence is defined by the condition that

$$\nabla^\Phi \mathbf{E}_a = 0 \tag{1a}$$

$$\mathbf{E}_a(X) = \underset{a}{e}^A(X)\partial_A \tag{1b}$$

where $X = (X^A; A = 1, 2, 3)$ is a coordinate system on $\mathscr{B}$. Thus, the ∇^Φ-covariantly constant metric tensor $\mathbf{g}_\Phi$ has the form

$$\mathbf{g}_\Phi(X) = g_{ab} E^a(X) \otimes E^b(X) \tag{2a}$$

$$g_{ab} = g_{ba} = \text{const} \tag{2b}$$

$$E^a(X) = \overset{a}{e}_A(X)\, dX^A, \qquad \underset{a}{e}^A(X)\overset{b}{e}_A(X) = \delta^b_a \tag{2c}$$

and defines an *internal length measurement* in the body.

The torsion tensor $\mathbf{S}[\Phi]$ of the teleparallel connection has the form

$$\begin{aligned} \mathbf{S}[\Phi] &= S^a{}_{bc}\mathbf{E}_a \otimes E^b \otimes E^c \\ S^a{}_{bc} &= \overset{a}{e}_A \underset{b}{e}^B \underset{c}{e}^C S^A{}_{BC}, \qquad S^A{}_{BC} = \Gamma^A_{[BC]}[\Phi] \end{aligned} \tag{3}$$

and can be introduced independently of this connection (Trzęsowski, 1987):

$$\begin{aligned} \mathbf{S}[\Phi] &= \tau = \mathbf{E}_a \otimes \tau^a \\ \tau^a &= dE^a = \tau^a{}_{bc} E^b \wedge E^c \\ [\mathbf{E}_a, \mathbf{E}_b] &= C^c_{ab}\mathbf{E}_c \\ C^c_{ab} &= 2S^c{}_{ab} = -\tau^c{}_{ab} \end{aligned} \tag{4}$$

where $[\mathbf{u}, \mathbf{v}] = \mathbf{u} \circ \mathbf{v} - \mathbf{v} \circ \mathbf{u}$ denotes the commutator (bracket) of vector fields $\mathbf{u}$ and $\mathbf{v}$ considered as first-order differential operators [e.g., equation (1b)]. The triple $\tau_\Phi = (\tau^a; a = 1, 2, 3)$ of 2-forms τ^a is called a *Burgers field* (Trzęsowski and Sławianowski, 1990), and $\mathbf{S}[\Phi]$ can be interpreted as a tensorial measure of the dislocation density (Section 3). It is easy to see that the global rescaling of the internal length measurement defined by

$$\begin{aligned} \Phi \to \Phi\mathbf{L} &= (\mathbf{E}_a L^a{}_b) \\ \mathbf{L} &= \|L^a{}_b; a, b = 1, 2, 3\| \in GL^+(3) \end{aligned} \tag{5}$$

where $GL^+(3)$ denotes the group of all real 3×3 matrices with positive determinant, does not change this tensorial measure, i.e., $\mathbf{S}[\Phi\mathbf{L}] = \mathbf{S}[\Phi]$

(Trzęsowski and Sławianowski, 1990), and therefore we can assume, without loss of generality, that

$$\begin{aligned} \mathbf{g}_\Phi = \mathbf{g} &= \delta_{ab} E^a \otimes E^b \\ &= g_{AB}\, dX^A \otimes dX^B \\ g_{AB} &= \delta_{ab} \overset{a}{e}_A \overset{b}{e}_B \end{aligned} \tag{6}$$

Finally, we come to the conclusion that the distribution of dislocations in a crystalline solid ought to be described by the Burgers field τ_Φ, considered as a geometric object on the material Riemannian space $(\mathscr{B}, \mathbf{g}_\Phi)$ associated with a global moving frame Φ [equation (6)], rather than by the teleparallel connection ∇^Φ only (Sections 2 and 3). If so, the field equations following from geometric identities can be formulated as balance equations on the Riemannian space (Section 4). This leads to the identification of source terms responsible for interactions of dislocations and point defects created by them (Section 4). Note that such an approach to the description of dislocations is consistent with any of those field theories that admit the pair $(\mathbf{g}_\Phi, \tau_\Phi)$ as elementary geometric objects describing simultaneous occurrence of dislocations and point defects in a crystalline solid. For example, this is the case of the gauge theory of dislocations based on the Riemann–Cartan connection corresponding to $\mathbf{g}_\Phi$ (i.e., the general connection metric with respect to $\mathbf{g}_\Phi$), and thus taking into account the existence of general distributions of point defects not disturbing the local homogeneity of continuously dislocated bodies (Trzęsowski, 1993).

We will use the so-called geometric frame references, i.e., coordinate systems $X = (X^A)$ such that $[X^A] = [dX^A] = [l]$, $[\partial_A = \partial/\partial X^A] = [l^{-1}]$, where $[l] = \text{cm}$ in the cgs units system.

2. CONTINUIZED CRYSTALS

Assume that a stress-free crystalline solid is loaded by boundary tractions in the elastic regime. The occurrence of crystalline structure defects can be recognized by observing that unloading does not take the body back to its original configuration. The unloaded state will thus contain a residual stress field. On the other hand, we assume that the stored energy is only due to elastic deformation and clearly residual stresses cannot be captured by a deformation gradient because these would model a body that unloads completely (Lagoudas and Edelen, 1989). In the case of dislocated monocrystalline solids we can characterize deformations of that unloaded state based on an assumption that the distorted lattice is uniquely defined everywhere (e.g., Bilby *et al.*, 1955; Kondo, 1955, 1962; Kondo and Yuki, 1958). Namely, following Kondo, one images removing a macroscopically small part of the dislocated body and allowing it to relax (by removing all boundary tractions) up to an unstrained state called the *natural state*. The discrete material structure of the natural state of a monocrystalline solid coincides with a perfect lattice. Let us consider, in order to describe the Kondo *gedanken* experiment explicitly, a reference configuration $\mathscr{B}_\kappa = \kappa(\mathscr{B})$, $\kappa: \mathscr{B} \to E^3$-diffeomorphism, being an open and connected subset of the three-dimensional Euclidean point space E^3 (configurational space of the body $\mathscr{B}$). We will consider a distinguished diffeomorphism κ called, as well as the image $\mathscr{B}_\kappa$ of the body, the *reference configuration* (of the body). Then, the body $\mathscr{B}$ and the coordinate system $X = (X^A)$ on it can be identified with $\mathscr{B}_\kappa$ and the coordinate system $X_\kappa = X \circ \kappa^{-1}$ on $\mathscr{B}_\kappa$, respectively. Let $X = X(P)$ denote Lagrange coordinates of a point $P \in \mathscr{B}$ $(= \mathscr{B}_\kappa \subset E^3)$ and let dX^A denote the distance between points $P, Q \in \mathscr{B}$. If $\delta x^i(X)$ denotes the distance between these material points in a deformed state $\lambda(\mathscr{B})$, $\lambda: \mathscr{B} \to E^3$-diffeomorphism in E^3, and $\delta\xi^a(X)$ is the same relaxed material element, then the relations

$$\begin{aligned} \delta x^i(X) &= F^i{}_A(X)\, dX^A \\ \delta\xi^a(X) &= P^a{}_A(X)\, dX^A \\ \delta x^i(X) &= B^i{}_a(X)\, \delta\xi^a(X) \end{aligned} \tag{7}$$

define the so-called *distortions*: total $(F^i{}_A)$, plastic $(P^a{}_A)$, and elastic $(B^i{}_a)$. It follows that

$$\begin{aligned} F^i{}_A(X) &= B^i{}_a(X) P^a{}_A(X) \\ F^i{}_A(X) &= \lambda^i_{,A}(X) \end{aligned} \tag{8}$$

where $(\lambda^i(X), i = 1, 2, 3)$ denotes a coordinate description of the deformation λ in Cartesian Eulerian coordinates (x^i) on E^3 and Lagrange coordinates (X^A) on $\mathscr{B}$ (Trzęsowski, 1993). Repeating the Kondo cutting-relaxation procedure for many macrosopically small elements of the body, we obtain the collection of relaxed material line elements $\delta\xi^a(X(P))$, $P \in \mathscr{B}$, defining a moving coframe $\Phi^* = (E^a; a = 1, 2, 3)$ by

$$E^a(X) = \delta\xi^a(X) \tag{9}$$

that is,

$$\begin{aligned} E^a(X) &= \overset{a}{e}_A(X)\, dX^A \\ \overset{a}{e}_A(X) &= P^a{}_A(X) \end{aligned} \tag{10}$$

The condition that the collection of relaxed line elements cannot be captured by a deformation gradient (of a global deformation of the reference configuration) means then that, for at least one 1-form E^a, one should have

$$\tau^a = dE^a \neq 0 \tag{11}$$

If lattice defects are absent, i.e., $\tau^a = 0$, $a = 1, 2, 3$, then the natural state can be obtained by a global deformation of the reference configuration or, equivalently, there exists a Cartesian Lagrange coordinate system $\xi = (\xi^a)$, $\xi^a = \xi^a(X)$, on $\mathscr{B}$ such that

$$\begin{aligned} E^a &= d\xi^a \\ \overset{a}{e}_A(X) &= \xi^a_{,A}(X) \end{aligned} \tag{12}$$

and in the internal length measurement metric tensor $\mathbf{g}$ [equation (6)] has the form

$$\begin{aligned} \mathbf{g} &= \delta_{ab}\, d\xi^a \otimes d\xi^b \\ g_{AB} &= \delta_{ab}\xi^a_{,A}\xi^b_{,B} \end{aligned} \tag{13}$$

The nonintegrability condition (11), considered as a representation of the mutual discrepancy of macroscopically small relaxed line elements $\delta\xi^a$, ought to be treated as a continuous limit neglecting the finiteness of the lattice spacing of a dislocated crystalline solid. We can think, for example, of some limiting process in which lattice constants of a Bravais lattice (describing the discrete material structure of the natural state) decreases more and more but the lattice rotational symmetries as well as the mass per unit volume and the content of defects remain unchanged. The resulting body, called a *continuized crystal* (Kröner, 1984, 1986), retains locally the most characteristic properties of the original crystal, namely the existence of three crystallographic directions at each point, the rotational equivalence of triads of these directions, and the existence of internal length measurement scales along these directions. Let us consider a moving frame $\Phi = (\mathbf{E}_a;\, a = 1, 2, 3)$ of base vectors parallel to the local crystallographic directions of a continuized crystal as the one defining the relaxed material line elements of the Kondo *gedanken* experiment according to equations (10) and (11) and the duality condition (2c). Then, from the concept of the continuized crystal there follows the existence of *local rotational uncertainty* to select the moving coframe $\Phi^* = (E^a;\, a = 1, 2, 3)$, i.e., 1-forms E^a are defined up to the transformation $E^a(X) \to E'^a(X)$, where

$$\begin{aligned} E'^a(X) &= Q^a{}_b(X)E^b(X) \\ \mathbf{Q} &= \|Q^a{}_b\|: \quad \mathscr{B} \to G \subset SO(3) \end{aligned} \tag{14}$$

where G is the material symmetry group of a (macroscopically) homogeneous crystalline solid and $SO(3)$ denotes the group of all proper 3×3

orthogonal matrices (Trzęsowski, 1993). The group G can be identified with the group of point symmetries of an ideal Bravais reference lattice (defining a discrete monocrystalline structure of the solid in its reference configuration $\mathscr{B}_\kappa$) or G can be identified with the group of symmetries of a crystal texture. The pair (Φ, G) is called a *Bravais moving frame* (Trzęsowski, 1993). Note that according to the identification (9) of 1-forms E^a with the relaxed material line elements $\delta\xi^a$, the internal length measurement metric tensor $\mathbf{g}$ [equation (6)] gives us information that dislocations have no influence on the local metric properties of a continuized crystal (see Section 1). Thus, the triple $(\Phi, G, \mathbf{g})$ represents the *short-range order* of a dislocated continuized crystal.

It ought to be stressed that the base vectors $\mathbf{E}_a$, $a = 1, 2, 3$, do not describe translational symmetries of an ideal local lattice (even in the case of a monocrystalline solid). This is because in a continuized crystal translational symmetries are lost and only rotational symmetries (of the considered crystalline material) are preserved. However, the vector fields $\mathbf{E}_a$ define (in like manner as in the case of a discrete monocrystalline structure) internal length measurement scales along local crystallographic directions. The relaxed line elements $\delta\xi^a(X)$, $X = X(P)$, $P \in \mathscr{B}_\kappa$, corresponding to these scales according to the identification (9) are translated and rotated with respect to one another and fail to mesh to form a Euclidean length measurement. Their *translational discrepancy* is described by the nonintegrability condition (11). Their *rotational discrepancy* is represented by (infinitesimal) relative rotations of local crystallographic directions and thus can be described by the so-called Ricci coefficients of rotation $\omega_c{}^a{}_b$ defined by (Trzęsowski, 1993)

$$\begin{aligned} \nabla^g \mathbf{E}_a &= \omega_a^b \otimes \mathbf{E}_b \\ \omega_b^a &= \omega_c{}^a{}_b E^c = -\omega_a^b \\ \boldsymbol{\omega}_c &= (\omega_c{}^a{}_b) \in so(3) \end{aligned} \tag{15}$$

where ω_a^b are connection 1-forms of the Levi-Civita covariant derivative ∇^g corresponding to the metric tensor $\mathbf{g}$, and $so(3)$ denotes the Lie algebra [of the Lie group $SO(3)$] consisting of 3×3 real antisymmetric matrices representing infinitesimal rotations.

3. DISLOCATION DENSITY TENSOR

The occurrence of dislocations breaks the long-range order of a crystalline solid. It manifests itself in the existence of different short-range orders in macroscopically small neighborhoods of different points of the continuized crystal (see Section 2), and can be quantitatively measured by the so-called Burgers vector corresponding to a (macroscopic) closed contour γ in $\mathscr{B}$. Let us consider, in order to formulate a definition of this vector, a family of local diffeomorphisms $\lambda = \{\lambda_P: U_P \to E^3; P \in \mathscr{B}\}$ where E^3 is the configurational three-dimensional Euclidean point space and $\{U_P, P \in \mathscr{B}\}$ is an open covering of the reference configuration $\mathscr{B} = \mathscr{B}_\kappa \subset E^3$ (see Section 2) such that

$$d\lambda_P(\mathbf{E}_a)(P) = \epsilon_a \in \vec{E}^3 \tag{16}$$

where (Φ, G), $\Phi = (\mathbf{E}_a)$ is the Bravais moving frame (Section 2), and $(\epsilon_a; a = 1, 2, 3)$ is an orthonormal base of the Euclidean vectorial space $\vec{E}^3$ of translations in E^3 corresponding to a Cartesian coordinate system (ξ^a) on E^3. Let γ be a closed contour in $\mathscr{B}$ passing by the point $P \in \mathscr{B}$ and with its tangent vector field $\dot{\gamma}$:

$$\begin{gathered} \gamma\colon \ \langle \alpha, \beta \rangle \to \mathscr{B}, \qquad \gamma(\alpha) = \gamma(\beta) = P \\ \dot{\gamma}(t) = \dot{\gamma}^a(t)\mathbf{E}_a(\gamma(t)) \in T_{\gamma(t)}(\mathscr{B}), \qquad t \in (\alpha, \beta) \end{gathered} \tag{17}$$

where $T_Q(B)$ $(\cong \vec{E}^3)$ denotes the space tangent to the differential manifold $\mathscr{B}$ in Q.

Denoting

$$\dot{\gamma}_\lambda(t) = \dot{\gamma}^a(t)\epsilon_a \in \vec{E}^3 \tag{18}$$

we can define the curve γ_λ in E^3 by

$$\begin{gathered} \gamma_\lambda\colon \ \langle \alpha, \beta \rangle \to E^3 \\ \overrightarrow{P_\lambda \gamma_\lambda(t)} = \int_\alpha^t \dot{\gamma}_\lambda(s)\, ds, \qquad P_\lambda = \lambda_P(P) = \lambda_P(\gamma(\alpha)) \end{gathered} \tag{19}$$

possessing $\dot{\gamma}_\lambda$ as its tangent vector field. The curve γ can be considered as a continuous counterpart of the so-called *Burgers circuit*—an atom-to-atom path in a crystal containing dislocations, which form a closed loop (e.g., Hull and Bacon, 1984). Then the curve γ_λ, being an unclosed contour in E^3, constitutes a counterpart of the same circuit in a perfect crystal. Consequently, a vectorial measure of this unclosing defined by

$$\mathbf{b}(\gamma) = \overrightarrow{P_\lambda \gamma_\lambda(\beta)} = b^a(\gamma)\epsilon_a \tag{20}$$

can be called the *Burgers vector* (like the one completing the circuit in a perfect crystal). Note that here the Burgers vector is a closure running from the start of the circuit to its finish; frequently, the opposite convention of the Burgers vector orientation is taken (e.g., Hull and Bacon, 1984). It follows from equations (17)–(20) that

$$b^a(\gamma) = \oint_\gamma E^a = \int_\alpha^\beta \dot{\gamma}^a(t)\, dt$$
$$[b^a(\gamma)] = [l], \qquad [l] = \text{cm} \tag{21}$$

Let us consider the reference configuration $\mathscr{B} = \mathscr{B}_\kappa$ (see Section 2) as a Riemannian manifold endowed with the internal length measurement metric tensor $\mathbf{g}$ [equation (6)], and let $\Sigma \subset \mathscr{B}$ be a two-dimensional compact and oriented Riemannian submanifold of $(\mathscr{B}, \mathbf{g})$ (a surface) possessing the closed contour γ as its boundary. Then Stoke's theorem states that (Von Westenholz, 1978):

$$b^a(\gamma) = \int_\Sigma \tau^a \tag{22}$$

where [see equations (3), (4), and (11)]

$$\tau^a = \tfrac{1}{2}\tau^a{}_{BC}\, dX^B \wedge dX^C$$
$$\tau^a{}_{BC} = \overset{b}{e}_B \overset{c}{e}_C \tau^a{}_{bc} = 2\partial_{[B}\overset{a}{e}_{C]} \tag{23}$$
$$[\tau^a] = [l], \qquad [\tau^a{}_{BC}] = [l^{-1}]$$

and

$$\int_\Sigma \tau^a = \tfrac{1}{2}\int_\Sigma \tau^a{}_{BC}\, dS^{BC}$$
$$dS^{BC} = l^{BC}\, dS, \qquad l^{BC} = e^{BCD} l_D \tag{24}$$
$$l_D = g_{DE} l^E, \qquad l_A l^A = 1, \qquad [l_D] = [1]$$

where dS, $[dS] = [l^2]$, denotes the surface element of Σ normal to the unit vector $\mathbf{l} = l^A \partial_A$ and [equation (6)]

$$e^{ABC} = g^{-1/2} \epsilon^{ABC} \tag{25a}$$

$$g = \det\|g_{AB}\| = e_\Phi^{-2}, \qquad e_\Phi = \det\|\overset{}{\underset{a}{e}}{}^A\| \tag{25b}$$

where ε^{ABC} denotes the permutation symbol. Introducing on the Riemannian manifold $(\mathscr{B}, \mathbf{g})$ the so-called *dislocation density tensor* $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha} = \alpha^{AB} \partial_A \otimes \partial_B$$
$$\alpha^{BA} = \tfrac{1}{2}\tau^A{}_{CD} e^{CDB}, \qquad \tau^A{}_{BC} = \underset{a}{e}{}^A \tau^a{}_{BC} \tag{26}$$
$$[\alpha^{AB}] = [l^{-1}], \qquad [\boldsymbol{\alpha}] = [l^{-3}]$$

and defining the vector field $\mathbf{b} = b^A \partial_A$ by

$$\rho b^A = l_D \alpha^{DA}$$
$$[\rho] = [l^{-2}], \qquad [b^A] = [l] \tag{27}$$

where ρ is a scalar independent of the choice of $\mathbf{l}$, we obtain that

$$b^a(\gamma) = \int_\Sigma \rho b^a \, dS$$
$$\mathbf{b} = b^a \mathbf{E}_a, \qquad b^a = \overset{a}{e}_A b^A \tag{28}$$

Therefore, the vector field $\mathbf{b}$ can be interpreted as a *local Burgers vector* corresponding to a dislocation line tangent to the $\mathbf{l}$ direction and ρ can be interpreted as the so-called *scalar density of dislocations*, i.e., the length of all dislocation lines included in the volume unit (De Witt, 1973). Note that here the volume element dV is Riemannian:

$$dV(X) = g(X)^{1/2} \, d^3X \tag{29}$$

where d^3X is the Euclidean volume element. Representing the tensor field $\boldsymbol{\alpha}$ in the form

$$\boldsymbol{\alpha} = \alpha^{ab} \mathbf{E}_a \otimes \mathbf{E}_b = \boldsymbol{\alpha}^a \otimes \mathbf{E}_a$$
$$\boldsymbol{\alpha}^a = \alpha^{ba} \mathbf{E}_b, \qquad \alpha^{ab} = \overset{a}{e}_A \overset{b}{e}_B \alpha^{AB} \tag{30}$$
$$[\boldsymbol{\alpha}^a] = [l^{-2}], \qquad [\mathbf{E}_a] = [l^{-1}]$$

we can rewrite (27) as

$$\rho \mathbf{b} = \mathbf{l} \boldsymbol{\alpha} \tag{31}$$

Let us observe that if we introduce the *Burgers covector* $\mathbf{b}^*(\gamma)$ by

$$\mathbf{b}^*(\gamma) = b_a(\gamma) \boldsymbol{\epsilon}^a$$
$$b_a(\gamma) = \delta_{ac} b^c(\gamma), \qquad (\boldsymbol{\epsilon}_a, \boldsymbol{\epsilon}^b) = \delta_a^b \tag{32}$$

where $(\cdot, \cdot)$ denotes the scalar product in the Euclidean vectorial space $\vec{E}^3$, then

$$b_a(\gamma) = \int_\Sigma \tau_a$$
$$\tau_a = \delta_{ab} \tau^b = \mathbf{E}_a \boldsymbol{\tau}, \qquad (\mathbf{E}_a, \mathbf{E}_b)_g = \delta_{ab} \tag{33}$$

where $(\cdot, \cdot)_g$ denotes the scalar product in the Riemannian manifold $(\mathscr{B}, \mathbf{g})$. Thus, the tensor field $\boldsymbol{\tau}$ [or, equivalently, the torsion tensor $\mathbf{S}[\Phi]$; equations (3) and (4)] can be also interpreted as a measure of the dislocation density. The tensorial measures $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ or the dislocation density can be mutually related by means of the Hodge operator $*$ on the Riemannian manifold

$(\mathscr{B}, \mathbf{g})$. Namely, it follows from equations (4) and (30) and from the definition of the Hodge operator (e.g., Von Westenholz, 1978) that

$$\tau^a = *\alpha^a \tag{34a}$$

$$\alpha^a = *\tau^a = \alpha^{ba} E_b, \qquad E_a = \delta_{ab} E^b \tag{34b}$$

where $[\alpha^a] = [1]$, and thus

$$\begin{aligned} \alpha^{ba} &= \tfrac{1}{2}\tau^a{}_{pq} e^{pqb} \\ \tau^a{}_{bc} &= \alpha^{pa} e_{pbc} \end{aligned} \tag{35}$$

where $e^{abc} \overset{*}{=} \varepsilon^{abc}$ [see equations (6) and (25)] is the permutation symbol. Introducing designations

$$\gamma^{ab} = \alpha^{(ab)}, \qquad t_a = \tau^b{}_{ba} \tag{36}$$

we obtain that

$$\begin{aligned} \alpha^{[ab]} &= \tfrac{1}{2} t_c e^{cab} \\ t_a &= e_{abc}\alpha^{bc} \end{aligned} \tag{37}$$

and

$$\begin{aligned} \tau^a{}_{bc} &= e_{bcd}\gamma^{da} - t_{[b}\delta^a_{c]} \\ [\gamma^{ab}] &= [t_a] = [l^{-1}] \end{aligned} \tag{38}$$

Note that there exists a field $\mathbf{Q} = \|Q^a{}_b\|: \mathscr{B} \to SO(3)$ of local rotations such that if

$$\begin{aligned} \Phi\mathbf{Q} &= (\mathbf{e}_a), \qquad \mathbf{e}_a = \mathbf{E}_b Q^b{}_a \\ \Phi^*\mathbf{Q} &= (e^a), \qquad e^a = Q_b{}^a E^b \end{aligned} \tag{39}$$

where $\|Q_a{}^b\| = \mathbf{Q}^T$, then

$$\begin{aligned} \gamma &= \gamma^{ab}\mathbf{E}_a \otimes \mathbf{E}_b = \lambda^a \mathbf{e}_a \otimes \mathbf{e}_a \\ t &= t_a E^a = \mu e^3, \qquad [\mu] = [\lambda^a] = [l^{-1}] \end{aligned} \tag{40}$$

and so, in the base $(\mathbf{e}_a)$,

$$\|\alpha^{ab}\| \overset{*}{=} \begin{Vmatrix} \lambda^1 & \kappa & 0 \\ -\kappa & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{Vmatrix}, \qquad \kappa = \frac{\mu}{2} \tag{41}$$

Let us consider a line in the Riemannian manifold $(\mathscr{B}, \mathbf{g})$, with its unit tangent vector field $\mathbf{l} = l^a \mathbf{E}_a$, as a dislocation line with its local Burgers vector $\mathbf{b} = b^a \mathbf{E}_a$, i.e., [see equations (30) and (31)]

$$\rho b^a = l_b \alpha^{ba}$$
$$l_a l^a = 1, \qquad l_a = \delta_{ab} l^b \tag{42}$$

A slip plane of a dislocation is defined as a plane containing both the dislocation line and Burgers vector of the dislocation (e.g., Hull and Bacon, 1984). So, a plane $\pi(\mathbf{l}, \mathbf{b})$ containing vectors $\mathbf{l}$ and $\mathbf{b}$ can be interpreted as a *local slip plane*. This means that if $\mathbf{n} = n^a \mathbf{E}_a \neq \mathbf{o}$ is a vector normal to $\pi(\mathbf{l}, \mathbf{b})$, then

$$b^a n_a = 0, \qquad l^a n_a = 0$$
$$n_a = \delta_{ab} n^b \tag{43}$$

For example, if [see equations (36) and (37)]

$$t_a = \mu n_a, \qquad n_a n^b = 1$$
$$\gamma^{ab} n_b = 0, \qquad l^a n_a = 0 \tag{44}$$

where μ is a scalar, then the condition (43) is fulfilled. It follows from equations (36) and (42) that

$$\rho b^a l_a = \gamma^{ab} l_a l_b \tag{45}$$

The considered line can be interpreted as an *edge dislocation* line if (Hull and Bacon, 1984).

$$b^a l_a = 0, \qquad b^a \neq 0 \tag{46}$$

or, equivalently, if

$$\gamma^{ab} l_a l_b = 0$$
$$l_a l^a = 1, \qquad b^a \neq 0 \tag{47}$$

A *screw dislocation* line can be defined by the condition

$$b^a = \eta l^a \tag{48}$$

or, equivalently, by the demand that $\mathbf{l}$ is a left eigenvector of the dislocation density tensor $\boldsymbol{\alpha}$ with an eigenvalue v:

$$l_b \alpha^{ba} = v l^a$$
$$l_a l^a = 1 \tag{49}$$

which means that

$$v = \rho \eta = \epsilon(\eta) b \rho = \gamma^{ab} l_a l_b$$
$$\epsilon(\eta) = \operatorname{sgn} \eta, \qquad \rho > 0, \quad b > 0 \tag{50}$$
$$\eta^2 = b^2 = (\mathbf{b}, \mathbf{b})_g = b^a b_a, \qquad l_a l^a = 1$$

The condition (50) admits screw dislocations with the same line sense but opposite Burgers vectors: they are the so-called right-handed screw if $\epsilon(\eta) = 1$ or left-handed screw if $\epsilon(\eta) = -1$, and these are also physical opposites of each other because they annihilate and restore a perfect crystal if brought together $[\epsilon(\eta) = 0]$ (Hull and Bacon, 1984). It follows from the above definitions that in the representation (41) of the dislocation density tensor $\boldsymbol{\alpha}$, diagonal elements correspond to screw dislocations whereas nondiagonal elements correspond to edge dislocations.

We see that while local slip planes are uniquely defined for edge dislocations, this is not the case for screw dislocations. However, if a screw dislocation lies in the slip plane of the edge dislocation, then the same plane can serve as a slip plane for the screw. From this we can proceed to the representation of a dislocation which is of mixed edge and screw character and lies along a path in the slip plane (Frank and Steeds, 1975). Namely, let us write the dislocation density tensor α^{ab} in the following form [see equations (36) and (37)]:

$$\begin{gathered}\alpha^{ab} = \gamma^{ab} + \beta^{ab} \\ \beta^{ab} = \tfrac{1}{2} t_c e^{cab} = -\beta^{ba}, \qquad \gamma^{ab} = \gamma^{ba}\end{gathered} \tag{51}$$

and let l^a denote an eigenvector of the symmetric tensor field γ^{ab} with its eigenvalue v, i.e.,

$$\begin{gathered}\gamma^{ab} l_b = v l^a \\ l_a l^b = 1, \qquad l_a = \delta_{ab} l^b\end{gathered} \tag{52}$$

The local Burgers vector b^a corresponding to l^a is defined by (42). If we define vectors $\underset{s}{b}{}^a$ and $\underset{e}{b}{}^a$ by

$$\begin{gathered}\rho_s \underset{s}{b}{}^a = l_b \gamma^{ba} \\ \rho_e \underset{e}{b}{}^a = l_b \beta^{ba}\end{gathered} \tag{53}$$

where ρ_e and ρ_s are positive scalars, then

$$\begin{gathered}\underset{e}{b}{}^a l_a = 0 \\ \underset{s}{b}{}^a = \eta l^a, \qquad \eta = v/\rho_s\end{gathered} \tag{54}$$

and

$$\begin{gathered}b^a = c_s \underset{s}{b}{}^a + c_e \underset{e}{b}{}^a, \qquad \underset{s}{b}{}^a \underset{e}{b}{}_a = 0 \\ c_s = \rho_s/\rho, \qquad c_e = \rho_e/\rho, \qquad c_s + c_e = 1\end{gathered} \tag{55}$$

It follows from (54) that a dislocation line possessing l^a as its unit tangent vector has the edge as well as screw character described by pairs

$$(\rho_e, \underset{e}{b}^a) \qquad \text{and} \qquad (\rho_s, \underset{s}{b}^a)$$

respectively. Thus, the same line has a mixed (edge and screw) character described by the pair (ρ, b^a), and

$$\rho b^a = \nu l^a + \mu m^a$$
$$m^a = \tfrac{1}{2} n_c l_b e^{cba}, \qquad t_c = \mu n_c \tag{56}$$
$$n^c n_c = 1, \qquad l_a m^a = 0, \qquad \mu > 0$$

If

$$l_a n^a = 0 \tag{57}$$

then $\pi(\mathbf{l}, \mathbf{b})$ is the local slip plane containing the direction $\mathbf{l}$ as well as the local Burgers vectors $\mathbf{b}_e$, $\mathbf{b}_s$, and $\mathbf{b}$.

4. BALANCE EQUATIONS

Let $\Omega \subset \mathscr{B}$ be a three-dimensional regular region with a regular closed boundary Σ. The vector field $\mathbf{F}(\Sigma)$ defined by [cf. equations (20)–(23)]

$$\mathbf{F}(\Sigma) = F^a(\Sigma)\boldsymbol{\epsilon}_a$$
$$F^a(\Sigma) = \int_\Sigma \tau^a \tag{58}$$

is called the *Frank vector* (Kadić and Edelen, 1983). It follows from equations (24), (30), (34), (58) and from the divergence theorem of Gauss (Von Westenholz, 1978) that

$$F^a(\Sigma) = \int_\Sigma (\boldsymbol{\alpha}^a, \mathbf{l})_g \, dS = \int_\Omega \operatorname{div}_g \boldsymbol{\alpha}^a \, dV \tag{59}$$

where dV denotes the Riemannian volume element of $(\mathscr{B}, \mathbf{g})$, dS denotes the surface element of Σ treated as a 2-dimensional submanifold of $(\mathscr{B}, \mathbf{g})$, $\mathbf{l}$ is the unit outer normal vector field on Σ, and

$$\operatorname{div}_g \boldsymbol{\alpha}^a = -\delta\alpha^a = \nabla^g_B \alpha^{Ba}$$
$$= g^{-1/2}\partial_B(g^{1/2}\alpha^{Ba}) \tag{60}$$
$$\boldsymbol{\alpha}^a = \alpha^{Aa}\partial_A, \qquad g = \det\|g_{AB}\|$$

where ∇^g denotes the Levi-Civita covariant derivative corresponding to $\mathbf{g}$, and δ denotes the codifferential operator on $(\mathscr{B}, \mathbf{g})$, i.e. (Von Westenholz, 1978)

$$\delta\alpha^a = -* d * \alpha^a \tag{61}$$

where $*$ denotes the Hodge operator. Since $d\tau^a = 0$ [see (4)] we obtain from equations (34), (60), and (61) that

$$\operatorname{div}_g \boldsymbol{\alpha}^a = 0 \tag{62}$$

i.e., the Frank vector vanishes. It is a conservation equation in the Riemannian space $(\mathscr{B}, g)$, equivalent to the following balance equation in the Euclidean space E^3 [see equations (25b) and (60)]:

$$\partial_B \alpha^{Ba} = j^a \tag{63a}$$

$$j^a = s_B \alpha^{Ba}, \qquad s_B = \partial_B \ln e_\Phi \tag{63b}$$

where it was assumed that $e_\Phi > 0$ (i.e., we consider an oriented Bravais moving frame). If **g** is a flat metric, then (63) reduces, in Cartesian Lagrange coordinate systems (X^A), to the following form:

$$\partial_B \alpha^{Ba} \overset{*}{=} 0 \tag{64}$$

which is usually interpreted as stating that lines of dislocations do not terminate within the crystal (Kondo, 1955; Kröner, 1960). Consequently, the case $j^a = 0$ means that dislocations must either form closed loops or branch into other dislocations (Hull and Bacon, 1984). However, in general the term j^a does not vanish, and thus this source term can be interpreted as representing the existence of dislocation lines terminating within the crystal. This is possible, e.g., due to the appearance of point defects created by dislocations (see Section 1) or due to the existence of grain boundaries in the considered dislocated macroscopically homogeneous body (see Section 2). The form of the source term j^a [equation (63b)] shows that it is produced by the variableness of the material volume element [equation (29)]. Consequently, we can distinguish *volume-preserving Bravais moving frames* defined by the condition that there exists a smooth field $\mathbf{L} = \|L^a{}_b\|: \mathscr{B} \to SL(3)$ of local transformations and an orthonormal base (ϵ_a) in the Euclidean space $\vec{E}^3$ such that [independently of the choice of the Lagrange coordinate system $X = (X^A)$] at each point of the body [identified with its distinguished reference configuration and thus with the identification $T_P(\mathscr{B}) \cong \vec{E}^3$, $P \in \mathscr{B}$; see Sections 2 and 3]

$$\mathbf{E}_a(X) L^a{}_b(X) = \boldsymbol{\varepsilon}_b \tag{65}$$

where $SL(3)$ denotes the group of 3×3 unimodular matrices. It follows from the condition (65) [and from equations (6), (25b) and (63)] that for the volume-preserving Bravais moving frame $e_\Phi \overset{*}{=} 1$, and thus $j^a \overset{*}{=} 0$ in Cartesian Lagrange coordinate systems $X = (X^A)$ although, in general, **g** is not a flat metric tensor. For example, a type of point defect called "point stacking fault" does not change the volume (Kröner, 1990).

It follows from equations (4), (6), and (15) that

$$\tau^a{}_{bc} = \omega_c{}^a{}_b - \omega_b{}^a{}_c \tag{66}$$

Therefore

$$t_a = \tau^c{}_{ca} = -\omega_c{}^c{}_a \tag{67}$$

and [see equation (63b)]

$$\begin{aligned} &\nabla^g_a E^a = t, \qquad t = t_a E^a \\ &t_a = -\nabla^g_A \underset{a}{e}{}^A = s_a - \partial_A \underset{a}{e}{}^A \\ &s_a = \underset{a}{e}{}^A s_A = \partial_a \ln e_\Phi \end{aligned} \tag{68}$$

where

$$\partial_a = \underset{a}{e}{}^A \partial_A$$

For example, if the Bravais moving frame (Φ, G) is volume-preserving, then (in Cartesian Lagrange coordinates)

$$\begin{aligned} &\partial_A \alpha^{Aa} \overset{*}{=} 0, \qquad t_a \overset{*}{=} -\partial_A \underset{a}{e}{}^A \\ &\|\underset{a}{e}{}^A\|: \quad \mathscr{B} \to SL(3) \end{aligned} \tag{69}$$

Note that from equations (38), (60), and (67) it follows that equation (62) also can be written in the following form:

$$\begin{aligned} &\partial_b \alpha^{ba} = 2q^a \\ &q^a = \boldsymbol{\alpha}^a \mathbf{t} = \gamma^{ab} t_b \end{aligned} \tag{70}$$

where $\mathbf{t} = t^a \mathbf{E}_a$, $t^a = \delta^{ab} t_b$. The vanishing of the field $\mathbf{q} = q^a \mathbf{E}_a$ means the existence of a distinguished distribution of local slip planes: namely those normal to the vector field $\mathbf{t}$ [see equations (42)–(45)]. For example, the vector field $\mathbf{q}$ vanishes if the distribution of dislocations is uniformly dense (i.e., $\tau^a{}_{bc} = \text{const}$) (Trzęsowski, 1987, 1992).

It follows from equations (1), (4), and (66)–(68) that the incompatibility of the distribution of relaxed macroscopically small material line elements (Section 2) has the curvature (of the Levi-Civita covariant derivative ∇^g) as well as the torsion (of the teleparallel covariant derivative ∇^Φ) character. Thus, from this point of view, the dislocation density tensor $\boldsymbol{\alpha}$ represents the torsion incompatibility related to the Riemannian volume element, and the balance equation (63) describes the influence of the curvature incompatibility on the distribution of dislocations. An equation describing the influence of a distribution of dislocations on the curvature incompatibility can be formulated with the help of the curvature identity for the Riemannian curvature tensor corresponding to the metric $\mathbf{g}$.

Namely, for the Riemannian curvature 2-form Ω^a_b we have the following representation (Trzęsowski, 1993):

$$\Omega^a_b = e^a{}_{bc} R^c$$
$$R^c = \tfrac{1}{2} R^c{}_{bd} E^b \wedge E^d, \qquad e^a{}_{bc} = \delta^{ap} e_{pbc} \tag{71}$$
$$R^c{}_{bd} = -\tfrac{1}{2} R_{bdpq} e^{pqc}, \qquad R_{abcd} = \delta_{dp} R_{abc}{}^p$$

where $R_{abc}{}^d$ are components [with respect to the moving frame $\Phi = (\mathbf{E}_a)$] of the Riemannian curvature tensor. Let us introduce the 1-forms Θ^a, $a = 1, 2, 3$ by [see equation (34b)]

$$\Theta^a = -\tfrac{1}{2} * R^a = \Theta^{ba} E_b \tag{72}$$

where

$$\begin{aligned} \Theta^{ba} &= \tfrac{1}{4} e^{bpq} R^a{}_{pq} \\ &= \tfrac{1}{4} e^{bpq} e^{ars} R_{pqrs} = \Theta^{ab} \end{aligned} \tag{73}$$

Thus

$$R_{abcd} = e_{abp} e_{cdq} \Theta^{pq} \tag{74}$$

and the correspondence between tensors Θ^{ab} and R_{abcd} is one-to-one. Moreover, the Ricci tensor R_{ab} has the form

$$\begin{aligned} R_{ab} = R_{cab}{}^c &= \Theta_{ab} - \Theta \delta_{ab} \\ \Theta_{ab} = \delta_{ac} \delta_{bd} \Theta^{cd}, &\qquad \Theta = \delta_{ab} \Theta^{ab} \end{aligned} \tag{75}$$

and the tensor Θ_{ab} covers with the so-called Einstein tensor:

$$\Theta_{ab} = R_{ab} - \tfrac{1}{2} R \delta_{ab} \tag{76a}$$

$$R = \delta^{ab} R_{ab} = -2\Theta \tag{76b}$$

From the curvature identity (Trzęsowski, 1993)

$$\nabla^g R^a = dR^a + \omega^a_b \wedge R^b = 0 \tag{77}$$

we obtain [taking into account (15)] that

$$\begin{aligned} * \, dR^a &= -\delta * R^a = \delta \Theta^a \\ \delta \Theta^a &= \omega_b{}^a{}_c \Theta^{bc} \end{aligned} \tag{78}$$

where

$$\delta \Theta^a = \nabla^g_B \Theta^{Ba} = -\mathrm{div}_g \, \boldsymbol{\vartheta}^a \tag{79}$$

and we have denoted

$$\boldsymbol{\vartheta}^a = \Theta^{ba} \mathbf{E}_b, \qquad \Theta^{Ba} = \underset{b}{e}{}^B \Theta^{ba} \tag{80}$$

Since the Levi-Civita covariant derivative $\nabla^g = (\omega^a_b)$ [see (15)] is torsion-free, i.e. (Trzęsowski, 1993)

$$dE^a + \omega^a_b \wedge E^b = 0 \tag{81}$$

we obtain that

$$\begin{aligned} \omega_b{}^a{}_c \Theta^{bc} &= \delta^{ad} \tau^p{}_{cd} \Theta^c{}_p \\ \Theta^a{}_b &= \delta_{bc} \Theta^{ac} \end{aligned} \tag{82}$$

and from equations (35), (78), and (79) we obtain the following balance equation on the Riemannian manifold $(\mathscr{B}, \mathbf{g})$:

$$\operatorname{div}_g \vartheta^a = \sigma^a \tag{83a}$$

$$\sigma^a = e^{abc} \Theta_{bd} \alpha_c{}^d, \qquad \alpha_b{}^a = \delta_{bc} \alpha^{ca} \tag{83b}$$

equivalent to the well-known identity for the Einstein tensor (Schouten, 1954)

$$\begin{aligned} \nabla^g_A \Theta^A{}_B &= g^{AC} \nabla^g_A \Theta_{CB} = 0 \\ \Theta^A{}_B &= \underset{a}{e}{}^A \overset{b}{e}_B \Theta^a{}_b \end{aligned} \tag{84}$$

It follows from (60) with α^{Ba} changed for Θ^{Ba} that (83) can be written also in the following form:

$$\begin{aligned} \partial_B \Theta^{Ba} &= r^a \\ r^a &= \sigma^a + s_B \Theta^{Ba} \end{aligned} \tag{85}$$

where s_B is given by equation (63b).

We see, comparing the form of source terms j^a and r^a appearing in equations (63) and (85), respectively, that the variableness of the (Riemannian) material volume element is the factor responsible for the mutual influence of the torsion and curvature incompatibilities [see remarks preceding (71)]. On the other hand, it is well known that the most important contribution of the interaction between a point defect and a dislocation is usually due to the distortion of the point defect produces in the surrounding crystal (e.g., Hull and Bacon, 1984). For vacancies or for interstitials located along local crystallographic directions (see Section 1) this distortion can be treated as possessing a spherical symmetry (Kröner, 1990). Therefore, in such particular cases, the variableness of the material volume element can be identified as a geometric factor responsible for interactions between dislocations and point defects created by them. However, the action of an interstitial can well be anisotropic, and then the above identification may be an approximation of the response law whose derivation requires energetic considerations and therefore goes beyond the presently employed differential geometry (Kröner, 1990). The discussed variableness of the material volume element can be dependent also, in the case of a dislocated macroscopically homogeneous body, on the existence of grain boundaries on the microlevel. However, the way these boundaries influence interactions between dislocations and point defects is not yet known. Note also that even in the case of a volume-preserving Bravais moving frame [equations (65) and (69)], dislocations influence the curvature incompatibility: namely through the source term σ^a [equations (83b) and (85) with $s_b = 0$].

5. FINAL REMARKS

If (Φ, G), $\Phi = (\mathbf{E}_a)$, is a Bravais moving frame, then the moving frame Φ is defined up to local rotations belonging to the material symmetry group $G \subset SO(3)$ of the considered crystalline solid [see (14)]. Therefore, a field theory of continuous distributions of dislocations should be invariant under the local action of the group G. For example, this invariance can be taken as a starting point of the gauge theory of continuously dislocated crystalline solids (Trzęsowski, 1993). Consequently, the existence of the conservation equation (84) for the Einstein tensor $\mathbf{\Theta}_\Phi = \Theta_{ab} E^a \otimes E^b$ suggests the physical importance of cases when the invariance condition

$$\begin{gathered} \mathbf{\Theta}_{\Phi \mathbf{Q}} = \mathbf{\Theta}_\Phi \\ \mathbf{Q}\colon \quad \mathscr{B} \rightarrow G \subset SO(3) \end{gathered} \tag{86}$$

is fulfilled. If G is a gauge group, then $G = SO(3)$ (three-parameter Lie group of Euclidean rotations) or $G = G(\mathbf{n})$ (one-parameter Abelian Lie group of all Euclidean rotations about a fixed axis parallel to a vector $\mathbf{n}$) (Trzęsowski, 1993). For such symmetry groups the invariance condition (86) means that [see (6)]

$$\begin{gathered} \mathbf{\Theta}_\Phi = \lambda_1 n \otimes n + \lambda_2 \mathbf{g}_\Phi \\ (\mathbf{n}, \mathbf{n})_g = n_a n^a = 1 \\ \mathbf{n} = n^a \mathbf{E}_a, \qquad n = n_a E^a, \qquad n_a = \delta_{ab} n^b \end{gathered} \tag{87}$$

where λ_1 and λ_2 are scalars for $G = G(\mathbf{n})$, and $\lambda_1 = 0$ for $G = SO(3)$. The Ricci tensor $\mathbf{R}_\Phi = R_{ab} E^a \otimes E^b$ defined by (75) has then the form

$$\begin{gathered} \mathbf{R}_\Phi = \lambda_1 n \otimes n + \lambda_3 \mathbf{g}_\Phi \\ \lambda_3 = \lambda_2 - \Theta \end{gathered} \tag{88}$$

where $\lambda_3 = \text{const}$ for $G = SO(3)$. Equation (84) reduces then to the condition $\lambda_2 = \text{const}$ if $G = SO(3)$ or, if $G = G(\mathbf{n})$ and $\mathbf{n}$ is a unit vector field, to the form

$$\partial_{\mathbf{n}}(\lambda_1 + \lambda_2) + \lambda_1 \operatorname{div}_g \mathbf{n} = 0$$
$$(\mathbf{n}, \mathbf{n})_g = n_A n^A = 1, \qquad \partial_{\mathbf{n}} = n^A \partial_A \tag{89}$$

The form (88) of the Ricci tensor $\mathbf{R}_\Phi$ can be associated with the existence of slip surfaces (i.e., surfaces with their tangent planes being local slip planes; see Section 3) invariant under the action of isometries of the material space $(\mathscr{B}, \mathbf{g}_\Phi)$. Namely, the following theorem (with slightly changed designations) is valid.

Theorem. (Bona and Coll, 1992). The necessary and sufficient condition for an internal length measurement metric tensor $\mathbf{g}_\Phi$ to admit a three-parameter Lie group G_3 acting on two-dimensional orbits as the maximal isometry group is that the Ricci tensor $\mathbf{R}_\Phi$ be of the form (88) with the unit eigenvector $\mathbf{n}$ of $\mathbf{R}_\Phi$ fulfilling the condition

$$\nabla^g n = \zeta(\mathbf{g}_\Phi - n \otimes n) \tag{90}$$

and satisfying

$$d(\lambda_1 + \lambda_3) \wedge n = d\lambda_3 \wedge n = 0 \tag{91}$$

where $d(\lambda_1 + \lambda_3)$ and $d\lambda_3$ do not vanish simultaneously. The orbits of the group G_3 action on $(\mathscr{B}, \mathbf{g}_\Phi)$ will then be constant-curvature surfaces orthogonal to the vector $\mathbf{n}$. Moreover, $\mathbf{n}$ must be geodesic.

It follows from equations (76b) and (88)–(90) that the factor ζ is given by

$$n^A \partial_A(\lambda_3 - \lambda_1) = 4\zeta\lambda_1 \tag{92}$$

and the 1-form n is closed:

$$dn = 0 \tag{93}$$

Thus, at least locally, $n = d\varphi$, i.e., the orbits of G_3 are (constant-curvature) surfaces of the form $\varphi = \text{const}$. Moreover, the group G_3 is locally isomorphic to $SO(3)$ (positive curvature surfaces), $SO(2, 1)$ (negative curvature surfaces), or $E(2)$ (zero curvature surfaces), which denote the 3-dimensional rotation group, the 3-dimensional Lorentz group, and the 2-dimensional Euclidean group, respectively. If, additionally, the condition (44) is fulfilled, then the orbits of the group G_3 action can be interpreted as constant-curvature slip surfaces in a transversally isotropic, continuously dislocated crystalline solid. In this case the balance equation (70) reduces to the condition

$$\partial_b \alpha^{ba} = 0 \tag{94}$$

For isotropic, continuously dislocated crystalline solids, the invariance condition (86) with $G = SO(3)$ should be considered. The Ricci tensor $\mathbf{R}_\Phi$

then takes the form (88) with $\lambda_1 = 0$ and $\lambda_3 = \text{const}$. It corresponds to the condition that $\mathbf{g}_\Phi$ admits a six-parameter isometry group G_6 and $G_6 \cong SO(4)$, $G_6 \cong E(3)$, or $G_6 \cong SO(3, 1)$ according to the conditions $\lambda_3 > 0$, $\lambda_3 = 0$, or $\lambda_3 < 0$ (Bona and Coll, 1992). Consequently, the material space $(\mathscr{B}, \mathbf{g}_\Phi)$ has a constant curvature K and thus there is such a conformally Euclidean space that there exists (at least locally) a Cartesian Lagrange coordinate system $X = (X^A)$ such that (Sikorski, 1972)

$$\begin{gathered} g_{AB} = \kappa^{-2}\delta_{AB} \\ \kappa = \kappa(r) = 1 + (K/4)r^2 > 0 \\ K = \lambda_3/2, \qquad r^2 = \delta_{AB}X^A X^B \end{gathered} \tag{95}$$

For example, a Bravais moving frame defined by

$$\underset{a}{e}^A(x) \stackrel{*}{=} \kappa(r(X))L^A{}_a(X), \qquad \|L^A{}_a(X)\| \in SO(3) \tag{96}$$

defines a material space of the constant scalar curvature K.

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